1. Theoretical preliminaries ($\hbar = 1$)

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle$$

$$i\frac{\partial}{\partial t}\left(U|\psi(0)\rangle\right) = HU|\psi(0)\rangle$$

$$i\dot{U}|\psi(0)\rangle = HU|\psi(0)\rangle$$

$$i\dot{U} = HU$$

$$|\psi(t)\rangle = U(t)|\psi(0)\rangle$$

$$|\psi(0)\rangle = U(0)|\psi(0)\rangle$$

$$\Rightarrow U(0) = \mathbb{I}$$

Additionally, $U(t) \equiv \underbrace{T}_{\text{time-ordering operator}} \exp\left(-i\int_0^t H(t')dt'\right)$

$[H(t_1), H(t_2)] \neq 0$ for $t_1 \neq t_2$

$$\begin{cases} i\dot{U} = HU \\ U(0) = \mathbb{I} \end{cases} \Rightarrow U(t) - U(0) = \int_0^t dt_1\, \dot{U}(t_1)$$

$$\Rightarrow U(t) = \mathbb{I} + \int_0^t dt_1\, \dot{U}(t_1)$$

$$= \mathbb{I} - i\int_0^t dt_1\, H(t_1)U(t_1)$$

$$= \mathbb{I} - i\int_0^t dt_1\, H(t_1)\left[\mathbb{I} - i\int_0^{t_1} dt_2\, H(t_2)U(t_2)\right]$$

$$= \mathbb{I} - i\int_0^t dt_1\, H(t_1) - \int_0^t dt_1 \int_0^{t_1} dt_2\, H(t_1)H(t_2)U(t_2)$$

$$= \mathbb{I} - i\int_0^t dt_1\, H(t_1) - \int_0^t dt_1 \int_0^{t_1} dt_2\, H(t_1)H(t_2) + \cdots$$

For sufficiently short times, the propagator can be approximated in lowest order :

Approximation ①

$$U \simeq \mathbb{I} - i\int_0^t dt_1 H(t_1)$$

$$\mathcal{H}_s|\phi_j\rangle = \omega_j|\phi_j\rangle \Rightarrow E_j = \omega_j$$

Consider $\begin{cases} \text{system } \mathcal{H}_s = \sum_j \omega_j |\phi_j\rangle\langle\phi_j| \\ \text{time-dependent external } \mathcal{H}_e = \sin(\omega t)\sum_{p\neq q} h_{pq}|\phi_p\rangle\langle\phi_q| \end{cases}$

describing the interaction with a laser- or microwave field

weak field : $|h_{pq}| \ll |\omega_i - \omega_j|$

$$(\mathcal{H}_s + \mathcal{H}_e)|\phi_q\rangle = \omega_j|\phi_q\rangle + \sin(\omega t)\sum_{p\neq q} h_{pq}|\phi_p\rangle$$

※ Interaction picture

$H(t) = \underbrace{H_0}_{\text{free}} + \underbrace{V(t)}_{\text{interaction}}$

$$\begin{cases} |\psi_I(t)\rangle = e^{iH_0t}|\psi_S(t)\rangle \quad (S: \text{Schrödinger}) \\ \hat{O}_I(t) = e^{iH_0t}\,\hat{O}_S\,e^{-iH_0t} \end{cases}$$

State evolution : $i\frac{d}{dt}|\psi_I(t)\rangle = V_I(t)|\psi_I(t)\rangle$, $V_I(t) = e^{iH_0t}V(t)e^{-iH_0t}$

Advantages: ① Simplifies Perturbation Theory: Only $V(t)$ drives the dynamics, making it easier to treat perturbatively

② Decouple Evolution : The free evolution is handled separately.

$$\tilde{H}_e = e^{iH_s t} H_e e^{-iH_s t} = e^{iH_s t} \sin(\omega t) \sum_{p\neq q} h_{pq} |\phi_p\rangle\langle\phi_q| e^{-iH_s t}$$

$$= \sin(\omega t) \sum_{p\neq q} h_{pq} e^{i\omega_p t} |\phi_p\rangle\langle\phi_q| e^{-i\omega_q t}$$

$$= \sum_{p\neq q} \sin(\omega t) e^{i(\omega_p-\omega_q)t} h_{pq} |\phi_p\rangle\langle\phi_q|$$

$$= \sum_{p\neq q} \frac{1}{2i} (e^{i\omega t} - e^{-i\omega t}) e^{i(\omega_p-\omega_q)t} h_{pq} |\phi_p\rangle\langle\phi_q|$$

$$= \sum_{p\neq q} \frac{1}{2i} \left(e^{i(\omega_p-\omega_q+\omega)t} - e^{i(\omega_p-\omega_q-\omega)t} \right) h_{pq} |\phi_p\rangle\langle\phi_q|$$

Approximation ②

If $|\omega_p-\omega_q \pm \omega| t \gg 1$, the propagator $U(t) \simeq \mathbb{I} - i\int_0^t dt_1 H(t_1)$ induces only negligible transitions between level $|\phi_p\rangle$ and $|\phi_q\rangle$

Derive: $i\int_0^t dt_1 \tilde{H}_e(t_1) = \frac{1}{2} \sum_{p\neq q} \left(\int_0^t dt_1\ e^{i(\omega_p-\omega_q+\omega)t_1} h_{pq} |\phi_p\rangle\langle\phi_q| - \int \cdots \right)$

$$= \frac{1}{2} \sum_{p\neq q} \left(\frac{1}{i(\omega_p-\omega_q+\omega)} e^{i(\omega_p-\omega_q+\omega)t_1}\Big|_0^t h_{pq} |\phi_p\rangle\langle\phi_q| - \frac{1}{i(\omega_p-\omega_q-\omega)} \cdots \right)$$

When $|\omega_p-\omega_q \pm \omega| \gg 1$, the dynamics between $|\phi_p\rangle$ and $|\phi_q\rangle$ are restricted. t?

2. Trapped ion Hamiltonian

theoretical idealisation ①

Assume a chain of ions trapped in a 1D harmonic trap.

Ions repel each other, they oscillate in collective modes.

For one ion and one oscillatory mode: → collective excitation frequency

$$H = \underbrace{\frac{\omega_0}{2}\sigma_z}_{\text{(internal) } E_1 - E_0 = \omega_0} + \underbrace{\omega_t a^\dagger a}_{\text{(motional)}}$$

(internal) $E_1 - E_0 = \omega_0$

(motional) energy of the oscillatory mode

The interaction between the ion and a laser:

$$H_I = \Omega_R\, \sigma_x \otimes \cos(\omega t - kx)$$

often neglected

For ions in a harmonic trap, position operator x can be written in terms of the equilibrium position x_0 and the normal mode operators:

$$\begin{cases} \sigma_+ = \dfrac{\sigma_x + i\sigma_y}{2} & \text{raising operators} \\ \sigma_- = \dfrac{\sigma_x - i\sigma_y}{2} & \text{lowering operators} \end{cases}$$

$$\Rightarrow \sigma_x = \sigma_+ + \sigma_-$$

$$x = x_0 + \sqrt{\frac{\hbar}{2m\omega_t}}(a + a^\dagger)$$

$$\xRightarrow{x_0 = 0} x = \sqrt{\frac{\hbar}{2m\omega_t}}(a + a^\dagger) = \frac{\eta}{k}(a + a^\dagger)$$

$$\Rightarrow H_I = \frac{\Omega_R}{2}(\sigma_+ + \sigma_-)\left(e^{i\omega t} e^{-i\eta(a+a^\dagger)} + e^{-i\omega t} e^{i\eta(a+a^\dagger)}\right)$$

In the interaction picture,

$$\tilde{H}_I = e^{i(\frac{\omega_0}{2}\sigma_z + \omega_t a^\dagger a)t} H_I\, e^{-i(\frac{\omega_0}{2}\sigma_z + \omega_t a^\dagger a)t}$$

$$= \frac{\Omega_R}{2}\left[(\sigma_+ e^{i\omega_0 t} + \sigma_- e^{-i\omega_0 t})\left(e^{i\omega t} e^{-i\eta(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})} + e^{-i\omega t} e^{i\eta(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})}\right)\right]$$

※ $[\sigma^+, \sigma_x] = \sigma^-$, $[\sigma^-, \sigma_x] = -\sigma^+$

$[\sigma^+, \sigma_y] = i\sigma^+$, $[\sigma^-, \sigma_y] = -i\sigma^-$

$[\sigma^+, \sigma_z] = -2\sigma^+$, $[\sigma^-, \sigma_z] = 2\sigma^-$

$\sigma^+\sigma_z - \sigma_z\sigma^+ = -2\sigma^+$

$\sigma^+\sigma_z = \sigma_z\sigma^+ - 2\sigma^+$

$$\Rightarrow e^{\omega_0\sigma_z/2}\sigma^+ e^{-\omega_0\sigma_z/2}$$
$$= e^{\omega_0\sigma_z/2} e^{-\frac{\omega_0}{2}(\sigma_z - 2)}\sigma^+$$
$$= \sigma^+ e^{\omega_0}$$

※ $[\hat{n}, a] = -a$, $[\hat{n}, a^\dagger] = a^\dagger$

$\hat{n}a = a\hat{n} - a$

$$\Rightarrow e^{\omega_t a^\dagger a} a\, e^{-\omega_t a^\dagger a} = a\, e^{\omega_t(a^\dagger a - 1)} e^{-\omega_t a^\dagger a}$$
$$= a\, e^{-\omega_t}$$

$\eta = k\sqrt{\dfrac{\hbar}{2m\omega_t}}$: Lamb-Dicke parameter, describing the strength of motion coupling

$$= \frac{\hbar k}{\sqrt{2m\hbar\omega_t}} = \frac{P_{photon}}{P_{phonon}} \approx \frac{1}{10}$$

P → momentum

approximation ②

In the Lamb Dicke regime, $\eta \ll 1$

$$e^{\pm i\eta(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})} \approx \mathbb{I} \pm i\eta(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})$$

$$\tilde{H}_I \approx \frac{\Omega_R}{2}\Big[\sigma_+ e^{i\omega_0 t}\Big(e^{i\omega t}\big(I - i\eta(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})\big) + e^{-i\omega t}\big(I + i\eta(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})\big)\Big)$$

$$+ \sigma_- e^{-i\omega_0 t}\Big(e^{i\omega t}\big(I - i\eta(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})\big) + e^{-i\omega t}\big(I + i\eta(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})\big)\Big)\Big]$$

$$\approx \frac{\Omega_R}{2}\Big[\sigma_+ e^{i\omega_0 t}(e^{i\omega t} + e^{-i\omega t}) - i\eta\sigma_+ e^{i\omega_0 t}\big((e^{i\omega t} - e^{-i\omega t})(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})\big)$$

$$+ \sigma_- e^{-i\omega_0 t}(e^{i\omega t} + e^{-i\omega t}) - i\eta\sigma_- e^{-i\omega_0 t}\big((e^{i\omega t} - e^{-i\omega t})(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})\big)\Big]$$

$$\approx \frac{\Omega_R}{2}\Big[\sigma_+ e^{-i\Delta t} + \sigma_+ e^{i(\omega_0+\omega)t} - i\eta\sigma_+(e^{i(\omega_0+\omega)t} - e^{-i\Delta t})(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})$$

$$+ \sigma_- e^{i\Delta t} + \sigma_- e^{-i(\omega_0+\omega)t} - i\eta\sigma_-(e^{i\Delta t} - e^{-i(\omega_0+\omega)t})(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})\Big]$$

$\Delta = \omega - \omega_0$: Laser detuning

① $\Delta = 0$, carrier transition

$$\tilde{H}_I \approx \frac{\Omega_R}{2}\Big[\sigma_+ + \sigma_+ e^{2i\omega_0 t} - i\eta\sigma_+(e^{2i\omega_0 t} - 1)(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})$$

$$+ \sigma_- + \sigma_- e^{-2i\omega_0 t} - i\eta\sigma_-(1 - e^{-2i\omega_0 t})(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})\Big]$$

Using rotating wave approximation (RWA) approximation 3

$$\tilde{H}_I \approx \frac{\Omega_R}{2}(\sigma_+ + \sigma_-) = \frac{\Omega_R}{2}\sigma_x \qquad \text{"}H_c\text{"}$$

※ RWA : simplify by neglecting fast oscillating terms. These terms average out to zero over time due to their high frequency relative to the timescale of the interaction.

e.g. $H_I' = g(\underbrace{e^{i(\omega-\omega_0)t}}_{\text{slowly varying term (if } \omega\approx\omega_0)} + \underbrace{e^{i(\omega+\omega_0)t}}_{\text{rapidly oscillating term}}) \approx g e^{i(\omega-\omega_0)t}$

② $\Delta = -\omega_t \Rightarrow \omega = \omega_0 - \omega_t$ red sideband

$$\tilde{H}_I \approx \frac{\Omega_R}{2}\Big[\sigma_+ e^{i\omega_t t} + \sigma_+ e^{i(2\omega_0-\omega_t)t} - i\eta\sigma_+(e^{i(2\omega_0-\omega_t)t} - e^{i\omega_t t})(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})$$

$$+ \sigma_- e^{-i\omega_t t} + \sigma_- e^{-i(2\omega_0-\omega_t)t} - i\eta\sigma_-(e^{-i\omega_t t} - e^{-i(2\omega_0-\omega_t)t})(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})\Big]$$

$$\approx \frac{\Omega_R}{2}\Big[\sigma_+ e^{i\omega_t t} + \sigma_+ e^{i(2\omega_0-\omega_t)t} + i\eta\sigma_+(a + a^\dagger e^{2i\omega_t t}) - i\eta\sigma_+ e^{i(2\omega_0-\omega_t)t}(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})$$

$$+ \sigma_- e^{-i\omega_t t} + \sigma_- e^{-i(2\omega_0-\omega_t)t} - i\eta\sigma_-(a e^{-2i\omega_t t} + a^\dagger) - i\eta\sigma_- e^{-i(2\omega_0-\omega_t)t}(a e^{-i\omega_t t} + a^\dagger e^{i\omega_t t})\Big]$$

$$\overset{\text{RWA}}{\Longrightarrow} \tilde{H}_I \approx -i\eta\frac{\Omega_R}{2}(\sigma_- a^\dagger - \sigma_+ a) \qquad \text{"}H_r\text{"}$$

③ $\delta = \omega_t \Rightarrow \omega = \omega_0 + \omega_t$ blue sideband

$\rightarrow \rightarrow \tilde{H}_I \approx -i\eta \frac{\Omega_R}{2}(\sigma_- a - \sigma_+ a^\dagger)$ "H_b"

※ Physical meaning

$|g\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $|e\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\sigma_z|g\rangle = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1|g\rangle$

$\sigma_x = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $\sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix}$, $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$\sigma_+ = |e\rangle\langle g|$ $(\sigma_x = |e\rangle\langle g| + |g\rangle\langle e|,\ \sigma_z = |g\rangle\langle g| - |e\rangle\langle e|)$

$\sigma_- = |g\rangle\langle e|$

Carrier transition ($\delta = 0$) : the laser induces a flip between the two internal states ($|g\rangle \leftrightarrow |e\rangle$) of the ion without affecting its motional state.

Red sideband ($\delta = -\omega_t$) : $\begin{cases} |g, n\rangle \rightarrow |e, n-1\rangle \\ \text{or} \\ |e, n\rangle \rightarrow |g, n+1\rangle \end{cases}$ with spontaneous emission

one up, one down

resolved sideband cooling

$\downarrow$

$|n\rangle \rightarrow |n-1\rangle \rightarrow \cdots \rightarrow |0\rangle$

motional ground state

Blue sideband ($\delta = \omega_t$) : $\begin{cases} |e, n\rangle \rightarrow |g, n-1\rangle \\ \text{or} \\ |g, n\rangle \rightarrow |e, n+1\rangle \end{cases}$

all up or all down

3. Realistic parameters and some of the experimental reality

$\omega_t \sim 1\,\mathrm{MHz}$ $\qquad$ M, Mega ; G, Giga ; T, Tera

At room temperature ($T = 300K$, $k_b \simeq 1.38 \times 10^{-23}\,J/K$, $\hbar = 1.05 \times 10^{-34}\,J \cdot s$), one thus has a typical phonon occupation of $\frac{k_b T}{\hbar \omega_t} \simeq 10^7$

For one single excitation $\frac{k_b T}{\hbar \omega_t} \simeq 10^0$

$$\Rightarrow \quad T = \frac{\hbar \omega_t}{k_b} \simeq 10^{-5}\,K$$

Therefore one has to employ Doppler-cooling and side-band cooling to cool the ions close to their ground state.

X. Examples of quantum gates

① Hadamard gate

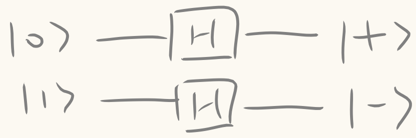
single-qubit gate, creates superposition

$$H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$H \cdot H = \mathbb{I}$



② Controlled phase gate (CP gate)

two-qubit gate, both qubits are control qubit, only if the control qubit is in the $|1\rangle$ state, CP gate applies a phase shift to the target qubit

$$U_{CP}(\phi) = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & e^{i\phi} \end{pmatrix} \xrightarrow{\phi = \pi} U_{CZ} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}$$

controlled-Z gate (CZ gate)

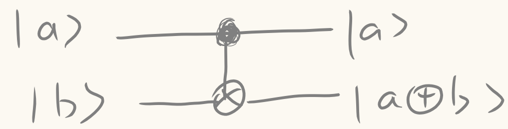
$|00\rangle \to |00\rangle$, $|01\rangle \to |01\rangle$,
$|10\rangle \to |10\rangle$, $|11\rangle \to e^{i\phi}|11\rangle$

③ Controlled-NOT gate (CNOT gate)

two-qubit gate, flips the target qubit only if the control qubit is in the $|1\rangle$ state

$$U_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \mathbb{I} & \\ & \sigma_x \end{pmatrix}$$

$|00\rangle \to |00\rangle$, $|01\rangle \to |01\rangle$,
$|10\rangle \to |11\rangle$, $|11\rangle \to |10\rangle$

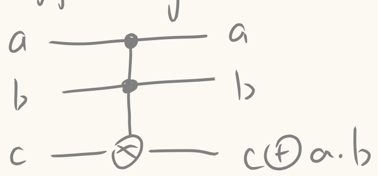


$$U_{CNOT} = (\mathbb{I} \otimes H)\, U_{CZ}\, (\mathbb{I} \otimes H)$$

Application: Bell state preparation:

$$|00\rangle \xrightarrow[\text{first qubit}]{H} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle) \xrightarrow{CNOT} \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

④ Toffoli gate (3 qubit gate)



4. Cirac - Zoller gates

A controlled-phase gate between two ions.

① "Write" the state of a qubit in the phonon mode

$$\underbrace{(\alpha|0\rangle + \beta|1\rangle)}_{\text{a qubit}} \otimes \underbrace{|0\rangle}_{\text{phonon mode}} \xrightarrow{\text{red-sideband transition}} |0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$$

requires that the motion is in ground state

② Direct interaction between qubit and phonon-mode

⇒ One can realise a controlled phase between qubit and phonon mode

Sequence:

$$|0\rangle_{Q_1} \otimes |0\rangle_{Q_2} \otimes |0\rangle_P = |000\rangle \xrightarrow{\text{write}} |000\rangle \xrightarrow{\text{c-phase}} |000\rangle \xrightarrow{\text{write}^{-1}} |000\rangle$$

$$|010\rangle \longrightarrow |010\rangle \longrightarrow |010\rangle \longrightarrow |010\rangle$$

$$|100\rangle \longrightarrow |001\rangle \longrightarrow |001\rangle \longrightarrow |100\rangle$$

$$|110\rangle \longrightarrow |011\rangle \longrightarrow -|011\rangle \longrightarrow -|110\rangle$$

Here $|1\rangle = \begin{pmatrix}1\\0\end{pmatrix} = |e\rangle$

swap the state of first qubit and phonon mode

Controlled phase gate with second qubit and phonon mode

5. Mølmer Sørensen gate

Consider n ions and a single phonon mode,

collective spin operators : $\Sigma_\pm = \sum_{i=1}^{n} \sigma_\pm^{(i)}$,

$$\sigma_\pm^{(i)} = \underbrace{I \otimes \cdots \otimes I}_{i-1} \otimes \sigma_\pm \otimes \underbrace{I \otimes \cdots \otimes I}_{n-i}$$

With 2 suitably detuned laser, one has the Hamiltonian in the interaction picture :

$$H_{ms} = i\eta\Omega(\Sigma_- a^\dagger e^{i\delta t} - \Sigma_+ a e^{-i\delta t}) + i\eta\Omega(\Sigma_- a e^{-i\delta t} - \Sigma_+ a^\dagger e^{i\delta t})$$

$$= i\eta\Omega(\Sigma_- - \Sigma_+)(a^\dagger e^{i\delta t} + a e^{-i\delta t})$$

At $T = \frac{2\pi}{\delta}$, $\int_0^T dt\, H(t)$ vanishes, so we need to consider the next higher order :

$$U \simeq I - i\int_0^t dt_1 H(t_1) - \int_0^t dt_1 \int_0^{t_1} dt_2 H(t_1) H(t_2)$$

$$\int_0^{2\pi/\delta} e^{i\delta t} dt = \frac{1}{i\delta} e^{i\delta t}\Big|_0^{2\pi/\delta} = \frac{1}{i\delta}(e^{i2\pi} - e^{i0}) = 0$$

Rewrite the propagator as :

$$U \simeq \exp\left[-i\int_0^t dt_1 H(t_1) - \frac{1}{2}\int_0^t dt_1 \int_0^{t_1} dt_2 \,[H(t_1), H(t_2)]\right]$$

always unitary

This is called Magnus expansion.

When $t = T$, $\int_0^T dt_1 \int_0^{t_1} dt_2\, [H(t_1), H(t_2)]$

$$\propto -\eta^2\Omega^2(\Sigma_- - \Sigma_+)^2 [a, a^\dagger] \int_0^T dt_1 \int_0^{t_1} dt_2 \sin\delta(t_1 - t_2)$$

$$\propto \frac{\eta^2\Omega^2}{\delta}(\Sigma_- - \Sigma_+)^2\, T$$

The propagator $U(T)$ is of the form as a propagator induced by the Hamiltonian $\sim (\Sigma_- - \Sigma_+)$